University of Wisconsin-Madison
Math 340 - Spring 2010
Linear Algebra
Final Exam

Exercise 1: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L((1,1)) = (3,0)$ and $L((-2,1)) = (1,1)$. Find $L((1,4))$.

Exercise 2: Some questions about the determinant.

(1) Let A be a $n \times n$ matrix satisfying $AA^t = I_n$. Prove that $\det A = 1$ or $\det A = -1$.
(2) Show that there is no 5×5 matrices A satisfying $A^t = -A$ (such a matrix is called skew symmetric).

Exercise 3: Prove or disprove that the following maps are linear.

(1) $L_1 : \mathbb{R}_2[X] \to \mathbb{R}$ with $L_1(a + bX + cX^2) = a + b - 2$
(2) $L_2 : \mathbb{R}_n[X] \to \mathbb{R}$, with $L_2(P) =< P, Q >$ where $Q \in \mathbb{R}_n[X]$ is fixed and $< P, Q >= \int_0^1 P(t)Q(t)dt$.
(3) $L_3 : \mathbb{R}^3 \to \mathbb{R}^3$, with $L_3((x, y, z)) = (x^2, y - z, 0)$.

Exercise 4: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map defined by

$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + 2y \\ x \end{pmatrix}$$

Let $\mathcal{C}$ be the canonical basis and consider $\mathcal{B} = \{v_1, v_2\}$ of $\mathbb{R}^2$ with,

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(1) Check that $\mathcal{B}$ is a basis of $\mathbb{R}^2$.
(2) (a) Find the matrix of L relative to the canonical basis $\mathcal{C}$, $[L]_{\mathcal{C}}^{\mathcal{C}}$.
 (b) Find the transition matrix, $[I]_{\mathcal{B}}^{\mathcal{C}}$, for changing from $\mathcal{B}$ to $\mathcal{C}$.
 (c) Find the transition matrix, $[I]_{\mathcal{C}}^{\mathcal{B}}$, for changing from $\mathcal{C}$ to $\mathcal{B}$.
 (d) Find the matrix of L relative to the basis $\mathcal{B}$, $[L]_{\mathcal{B}}^{\mathcal{B}}$.
(3) Consider the sequence of numbers u_n satisyfying $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + 2u_n$. (we have 1, 1, 3, 5, 11, 21, 43...)
 (a) Find a matrix A such that $\begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix} = A \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$.
 (b) Find a relation between $\begin{pmatrix} u_{n+2} \\ u_{n+1} \end{pmatrix}$, A and $\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$.
 (c) compute u_{102}.

Exercise 5: Let A be an $n \times n$ matrix. The goal of this exercise is to prove that $KerA = ImA$ if and only if $2rank(A) = n$ and $A^2 = 0$.

(1) Assume that $KerA = ImA$.
   (a) Prove that $2rank(A) = n$.
   (b) Prove that for all $X \in \mathbb{R}^n$, $A^2X = 0$. Deduce that $A^2 = 0$.
(2) Assume that $2rank(A) = n$ and $A^2 = 0$.
   (a) Prove that $\mathrm{Im}A \subseteq \mathrm{Ker}A$
   (b) Deduce that $\mathrm{Ker}A = \mathrm{Im}A$ by comparing their dimension.

Exercise 6: Let $L : \mathbb{R}_3[X] \to \mathbb{R}_2[X]$ be the linear map defined by

$$L(a + b + cX^2 + dX^3) = b - a + (c + d)X + aX^2.$$

Consider the basis $\mathcal{C}_3 = \{1, X, X^2, X^3\}$ and $\mathcal{C}_2 = \{1, X, X^2\}$.

(1) Find the matrix $[L]_{\mathcal{C}_3}^{\mathcal{C}_2}$ of L relative to $\mathcal{C}_3$ and $\mathcal{C}_2$.
(2) Find a basis for $\ker L$.
(3) Find $\mathrm{Im}L$.

Exercise 7: Let $(V, <, >)$ be an inner product space (of finite dimension) and let W be a subspace of V. Let $proj_W : V \to W$ be the orthogonal projection on W.

(1) Prove that $proj_W$ is linear (hint: take $\lambda \in \mathbb{R}$, and $v, v' \in V$ and find the orthogonal decomposition of $\lambda v + v'$).
(2) Prove that $\mathrm{Im}\ proj_W = W$.
(3) Prove that $\ker\ proj_W = W^\perp$ (hint: you can use the rank theorem).